Mathematics Investigation: How many ways can a number of telephones be connected if a telephone can't connect to more than one other?

Candidate Number Here

Contents

1 Introduction **3**
1.1 Purpose of exploration . . . 3
1.2 Background . . . 3

2 Graph theory **4**
2.1 Understanding graph theory . . . 4
2.2 Representing telephones with graphs . . . 4

3 Finding a solution with recursive programming **5**
3.1 Analysing the problem . . . 5
3.2 Recurrence relation formula . . . 9

4 Solution using differential calculus **9**

5 Extension: finding the maximum number of connections which can be made with n telephones **10**

6 Conclusion **14**

7 Bibliography **15**

1 Introduction

1.1 Purpose of exploration

I have found the idea of using math to find combinations in a given scenario very thought-provoking, as it initially seemed to me as though it would be impossible to derive any sort of method to find the number of ways in which something could be done. Of course, depending on the scenario, some ways of deriving this value will be harder than others and some will require the use of recursion – in which the formula to obtain a value of an input must rely on the result of the same formula of a different input.

This very idea of combinations has encouraged me to apply it to the scenario of connection telephones. Thus, the research question: **"How many ways can a number of telephones be connected if a telephone can't connect to more than one other?"**

The exploration will look at graph theory as a way to illustrate the problem, look into my solution of solving the problem and then look at an alternative solution to solving the problem using differential calculus. Then, as an extension, the exploration will look at finding the maximum number of connections which can be made and how many ways this can occur.

1.2 Background

Back before mobile phones were created, landlines were the common method of calling out friends and family members. These landlines were connected by metal wires or fibre-optic cables and allowed us to only call one person at a time. Nowadays, of course, we can talk to multiple people simultaneously in a group chat. Given that a connection can only be established between two phones, there must be a number of connection combinations that can be made with a given number of phones.

For example:

Given 2 phones (**A** and **B**), there are 2 ways in which they can be connected:
1. **A** and **B** are not connected
2. **A** and **B** are connected

Note that we include the situation in which they are not connected as a combination.

Given 3 telephones (**A**, **B** and **C**), there are 4 ways:
1. **A**, **B** and **C** are not connected
2. **A** and **B** are connected
3. **B** and **C** are connected
4. **C** and **A** are connected

Because we are taking into account that a phone can't connect to more than one other phone, we can't include all of the three phones connected as a combination.

In order to derive a formula for the number of ways in which the telephones can be connected, it is important to understand *graph theory* and to see a pattern when more telephones are added.

2 Graph theory

2.1 Understanding graph theory

Graph theory refers to a way in which data can be represented. A graph contains *vertices* and *edges*. In this case, a vertex will represent a telephone and an edge will represent a connection between two telephones. Note that an edge can only connect two vertices, which makes graph theory an excellent method of modelling this problem.

Since a telephone can't connect to more than one other telephone, we know that a graph of telephones can't be complete (that is, every vertex can't have an edge) and that each vertex can only have a degree of 0 or 1.

2.2 Representing telephones with graphs

The diagrams below show graph representations of the different connections which can be made from telephones 1-4. As stated, 0 telephones will have 1 combination.

1 telephone [1 combination]:

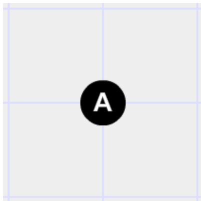
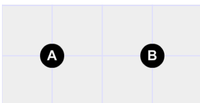
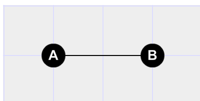


Figure 1: Graph A

2 telephones [2 combinations]:



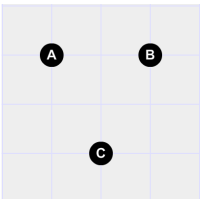
Graph AB 1



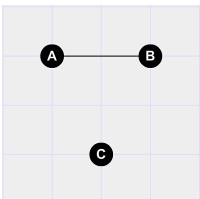
Graph AB 2

Figure 2: Graphs AB

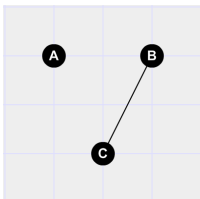
3 telephones [4 combinations]:



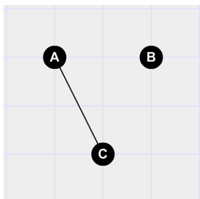
Graph ABC 1



Graph ABC 2



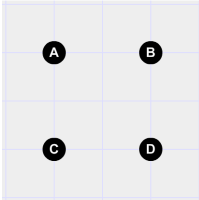
Graph ABC 3



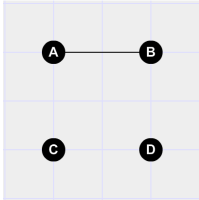
Graph ABC 4

Figure 3: Graphs ABC

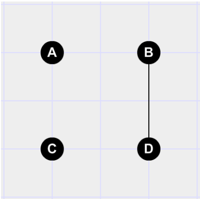
4 telephones [10 combinations]:



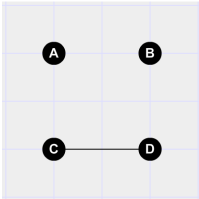
Graph ABCD 1



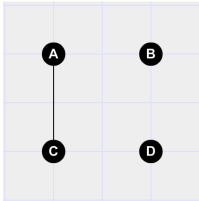
Graph ABCD 2



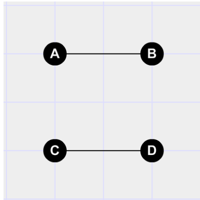
Graph ABCD 3



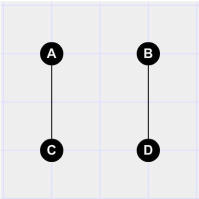
Graph ABCD 4



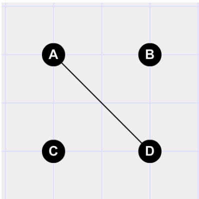
Graph ABCD 5



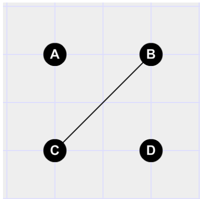
Graph ABCD 6



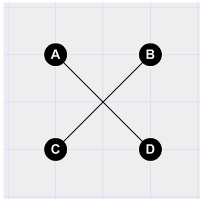
Graph ABCD 7



Graph ABCD 8



Graph ABCD 9



Graph ABCD 10

Figure 4: Graphs ABCD

An interesting thing to know is that the connection combinations are **bipartite graphs**. This means that the vertices can be put into two groups where the vertices are disjoint between the two groups. The next section will look at my solution to solving the problem.

3 Finding a solution with recursive programming

I have had an interest in programming since I was 14 years old. I gained a great deal of experience within a short amount of time, purely because of my keen interest in problem-solving and software-developing. Given this and that Computer Science is an interesting branch of Mathematics (which is also relatively new), I feel that it is only right to put programming to use in order to find a solution to the problem.

The next few sections will focus on analysing the problem and gradually adding *program code* to what will be the solution.

3.1 Analysing the problem

A possible solution to the problem is to deduce the total number of edge combinations possible for a graph of n vertices and count the number of solutions which satisfy this problem. However, this solution would be a waste of time and would also not give an exact formula which can be used to get the number of combinations for any integer n. What will be done instead is an analysis of the problem to see a pattern as n is increased.

1. **Consider 2 people (A and B) who each have a telephone.** We know that there can be 2 combinations for this case: either they are connected or they are not.

2. **Consider 3 people (A, B and C) who each have a telephone.**

- Firstly, we know that there must be combinations included where a person isn't connected. Let this be person C. So A and B have a combination: **2** (either they are connected or they are not). We will include this for the number of combinations for 3 people.

- Now consider the situation where C is connected to someone. C will have a choice between **2** people to connect to. Then when the person is connected to that one person, the other disconnected person will have only **1** connection combination: they must remain disconnected.

- Therefore, we multiply the values of **2** (from the choices of C) and **1** to make **2**. This is known as the Product Principle (AND rule); we must say that C connects to someone AND the other person connects to someone.

- Then we must add this multiplied value with the number of ways A and B can connect. $2 + 2 = 4$. This is known as the Addition Principle (OR rule).

- We must say that A and B are connected with C disconnected OR C is connected AND the other person is connected.

- Therefore, there are **4** ways that 3 telephones can connect to each other.

3. **Consider 4 people (A, B, C and D) who each have a telephone.**

- Using the same process as in the previous example with 3 people, consider the situation where D isn't connected to anyone. There are **4** ways that A, B and C can connect to each other.

- Now consider the situation where D is connected to another person. There are **3** people for D to choose from.

- When D is connected to someone else, there are 2 other people who are not connected. They can connect 2 ways (as seen above).

- Therefore, applying the Product Principle and the Addition Principle: $4+3(2) = 4+6 =$ **10**. This is, indeed, the correct number of ways in which 4 telephones can connect to each other.

4. **Following the examples above, a general solution for n telephones will be:** the combination of $n - 1$ telephones + the product of $(n - 1)$ telephones and the combination of $n - 2$ telephones, where the combination of 0 and 1 telephones is 1. To illustrate this, a recursive *Python* program which calculates the number of combinations is shown below:

```
def T(n):
    if n == 0:
        return 1
    else:
        return T(n-1) + (n-1) * T(n-2)
```

This code has a minor efficiency error: it recursively invokes the function T twice (once in $T(n-1)$ and another in $T(n-2)$). A solution to this issue is shown below:

```
def T(n):
    if n == 0:
        return [1, 1]
    else:
        combinations = T(n-1)
        return [combinations[0] + (n-1) * combinations[1],
            combinations[0]]
```

This solution makes use of only one recursive invocation of the function T and instead returns a pair of values. The first value of the pair is $T(n)$ and the second value is $T(n-1)$. Effectively, this will result in the function returning $T(n-1) + (n-1)T(n-2)$. Listed below are values obtained from 0 to 20 telephones using the program above:

Table 1: **Number of telephones with the number of possible connection combinations**

Number of Telephones	Number of Combinations
0	1
1	1
2	2
3	4
4	10
5	26
6	76
7	232
8	764
9	2620
10	9496
11	35696
12	140152
13	568504
14	2390480
15	10349536
16	46206736
17	211799312
18	997313824
19	4809701440
20	23758664096

In the above table (apart from the first two rows), let n be the number of telephones and let C_n be the number of combinations with n telephones. For example, $n = 0$, $C_0 = 1$ and $n = 1$, $C_1 = 1$. Apart from the first row and the second row, the combinations of n telephones are given by: $C_n = C_{n-1} + (n-1)C_{n-2}$, where C_{n-1} is the row above and C_{n-2} is two rows above. For example, $C_5 = C_{5-1} + (5-1)C_{5-2} = C_4 + 4C_3$. We can see from the table that $C_4 = 10$ and $C_3 = 4$. Therefore, $C_5 = 10 + 4(4) = 26$, which is, indeed, what is shown in the row in which $n = 5$. The relation we have used, $C_n = C_{n-1} + (n-1)C_{n-2}$, is known as a *Recurrence Relation* and it is discussed in the section below.

The table shows a large increase in combinations as the number of telephones is increased. It is quite easy to see why this is the case, as the relation above shows that we must add the previous number of combinations with the previous of the previous number of combinations multiplied by $(n-1)$. Therefore, it is obvious why it would gives a huge increase in the number of combinations when a telephone is added. A graph of this relation is shown below to illustrate the large rate of change for the relation:

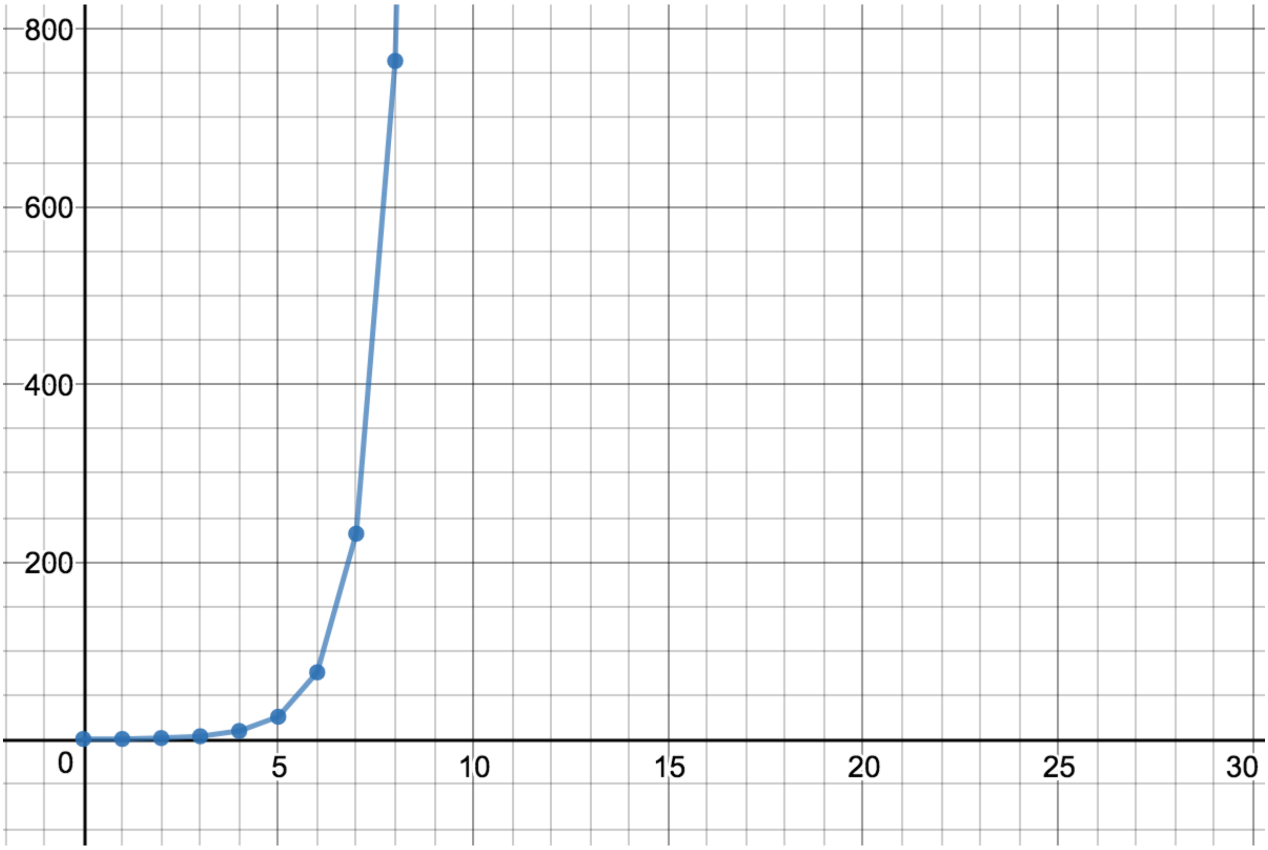


Figure 5: Graph to show number of telephones (x-axis) against number of connection combinations (y-axis)

The shape of the graph is similar to that of the factorial function: $f(x) = x!, x \geq 0$. The two graphs are shown below:

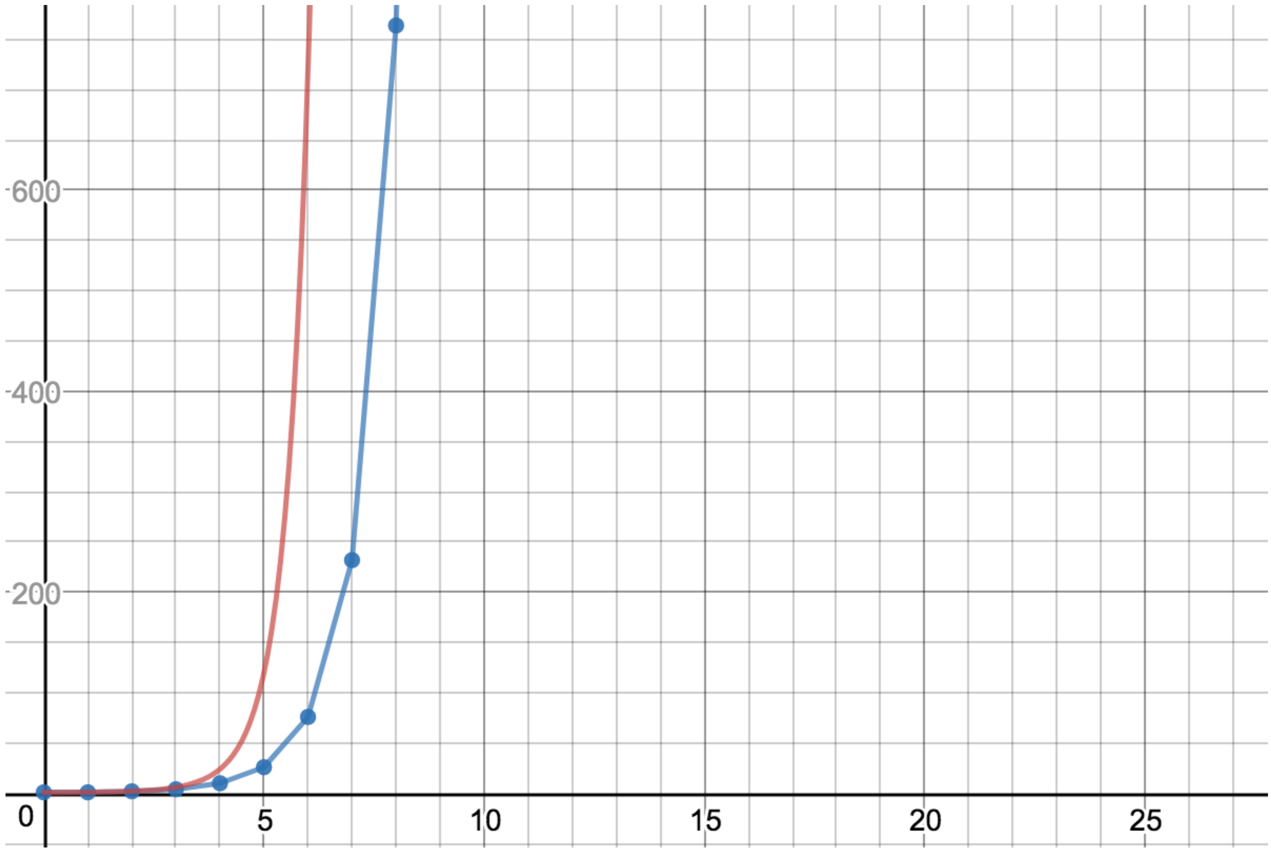


Figure 6: Graphs of telephone connection combinations relation (blue) and factorial function (red)

It would seem that the factorial function's rate of change is higher than that of the telephone connection combinations relation. This is the case for all $x \geq 0$, so the graphs never intersect. This is because the factorial function, $x!$ multiplies the previous term, $(x-1)!$ by x. Whereas, the telephone relation multiplies the previous term of the previous term by $(n-1)$. This explains why they get relatively close for large values of x, but never intersect.

3.2 Recurrence relation formula

As recursion is being used, an alternative form of the formula is a second-order recurrence relation: $t_n = t_{n-1} + (n-1)t_{n-2}$, and $t_0 = t_1 = 1$, as seen in the previous section. As with the program, the recurrence relation will also give the number of connection combinations given n telephones. I have tried coming up with a solution for the recurrence relation in terms of n. However, I was unsuccessful in doing so. After some research, I have found out that there is no solution to this recurrence relation because one term of the relation is dependent of n.

4 Solution using differential calculus

While researching, I came across a calculus solution to the problem by finding the n'th derivative of a function. Initially, I thought that what I was reading was false, as generally a derivative is supposed to "decrease" a function. However, after looking into it, I was astounded to find out that this solution actually works.

The formula is defined as: $f(x) = \mathrm{e}^{\frac{x^2}{2}+x}$ [1] and the value of the number of connection combinations of n telephones is: $\frac{d^n}{dx^n} f(0)$. The first 5 derivatives of $f(x)$ and their values at $x = 0$ are shown in the table below:

Table 2: **Derivatives of $f(x)$ and their values at $x = 0$**

Derivative order (n)	**Derivative**	**Value at $x = 0$**
1	$(x+1)\,\mathrm{e}^{\frac{x^2}{2}+x}$	1
2	$(x^2+2x+2)\,\mathrm{e}^{\frac{x^2}{2}+x}$	2
3	$(x^3+3x^2+6x+4)\,\mathrm{e}^{\frac{x^2}{2}+x}$	4
4	$(x^4+4x^3+12x^2+16x+10)\,\mathrm{e}^{\frac{x^2}{2}+x}$	10
5	$(x^5+5x^4+20x^3+40x^2+50x+26)\,\mathrm{e}^{\frac{x^2}{2}+x}$	26

It's very interesting to see that a polynomial of n'th degree is formed as the function is differentiated, with the constant term being the number of connection combinations. As can be seen, the number of terms in the polynomial formed increases for each derivative order. However, since we are setting $x = 0$ for the derivative, all terms in the polynomial with x result to 0 and $\mathrm{e}^{\frac{x^2}{2}+x}$ results to 1. This, of course, only leaves the constant term.

Given this is the case, it may not be entirely practical for us to use this method for large values of n, since x-terms result in either 0 or 1 to leave the constant term. However, it is still very interesting to know that calculus can still be used to solve the problem and gives us an opportunity to explore further into why this is the case and to learn more about how the two topics of calculus and discrete math can be related.

5 Extension: finding the maximum number of connections which can be made with n telephones

Before concluding, it would be interesting to think about the maximum number of connections which can be made, given a number of telephones (that is, the maximum number of edges which can exist for the a graph of n vertices to solve the problem).

- **Consider** 4 **vertices**. An edge connects two vertices and no vertex can have a degree of more than 1. Therefore, the maximum number of connections (edges) we can have is 2.
- **Consider** 5 **vertices**. Connecting 4 vertices can be done with 2 edges (as shown above), leaving 1 vertex which can't connect to another. Therefore, the number of connections we can have is also 2.
- **Consider** 6 **vertices**. Connecting 6 vertices can be done with 3 edges, leaving no vertices.. Therefore, the number of connections we can have is 3.

[1]The Telephone Numbers – Graph Theory. IB Maths Resources from British International School Phuket, IB Maths Resources, 10 June 2014, `ibmathsresources.com/2014/06/25/the-telephone-numbers-graph-theory/`

- **Consider n vertices, where n is even**. Connecting the n vertices can be done with $\frac{n}{2}$ edges (since each edge connects 2 vertices). Therefore, the number of connections we can have is $\frac{n}{2}$.

- **Consider n vertices, where n is odd**. Connecting $n-1$ vertices can be done with $\frac{n-1}{2}$ edges, leaving 1 vertex unconnected. Therefore, the number of connections we can have is $\frac{n-1}{2}$.

- **In general**, the number of connections we can have with n, where n is even or odd, is $\lfloor \frac{n}{2} \rfloor$.

We can also consider the number of ways in which we can connect the maximum number of edges. We have derived that, for n telephones, the maximum number of connections which can be made is $\lfloor \frac{n}{2} \rfloor$. From here, we must now count the number of ways we can select all pairs from n telephones. This is shown below:

- **Consider 2 telephones**. The maximum number of connections (edges) which can be made which still satisfies the problem is 1 (only 1 edge can connect the 2 vertices). There is only 1 way of doing this (since there are 2 vertices). Therefore, the number of ways will be 1.

- **Consider 4 telephones**. The maximum number of connections which can be made is 2 (using the formula from before). We must keep choosing a pair until there are none left. We can use the Addition Principle (AND rule) and the combination formula: $\binom{n}{r}$ to solve this problem. Choose 2 out of the 4 AND choose 2 out of the remaining 2: $\binom{4}{2}\binom{2}{2} = 6$. However, this counts the number of pairings 2! times (factorial of the maximum number of connections). Therefore, we must divide our result by 2: $\frac{6}{2} = 3$ ways.

- **Consider n telephones, where n is even**. The maximum number of connections which can be made is $\frac{n}{2}$. As done with 4 telephones: $\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}...\binom{2}{2}$. Evaluating this gives:

  $$\frac{n!}{2(n-2)!}\frac{(n-2)!}{2(n-4)!}...\frac{2}{2} = \frac{n(n-1)}{2}\frac{(n-2)(n-3)}{2}...\frac{2}{2} = \frac{n!}{2^{\frac{n}{2}}}.$$

  This counts the number of pairings $\frac{n}{2}!$ times. Therefore, dividing our result with this gives: $\frac{n!}{2^{\frac{n}{2}}\frac{n}{2}!}$ ways for an even number of telephones, n.

- **Consider n telephones, where n is odd**. The maximum number of connections which can be made is $\lfloor \frac{n}{2} \rfloor$. For an odd number, n, the number of combinations which can be made will be the combinations for $n-1$ (which is even), but then taking into account the number of ways which the remaining telephone can be connected (as not all telephones can be paired). Taking the formula for $n-1$, this will simply require the multiplication of the formula by n:

  $$n\frac{(n-1)!}{2^{\frac{n-1}{2}}\frac{n-1}{2}!} = \frac{n!}{2^{\frac{n-1}{2}}\frac{n-1}{2}!}$$ ways for an odd number of telephones.

- **In general**, the maximum number of connections which can be made with n telephones, where n is odd or even, is: $\dfrac{n!}{2^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{n}{2} \rfloor !}$. By flooring (rounding down) $\frac{n}{2}$, there is no need to have to write two formulae for the odd and even cases of n.

The values of $\dfrac{n!}{2^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{n}{2} \rfloor !}$ for values of n from 0-20 are shown below:

Table 3: **Number of telephones (n) with maximum connection combinations**

Number of telephones (n)	**Number of combinations**
0	1
1	1
2	1
3	3
4	3
5	15
6	15
7	105
8	105
9	945
10	945
11	10395
12	10395
13	135135
14	135135
15	2027025
16	2027025
17	34459425
18	34459425
19	654729075
20	654729075

The first obvious thing which can be seen here is that, for $n > 0$, numbers of combination are in pairs such that n and $n + 1$ have the same number of combinations where n is odd. To prove this works for all odd $n \in \mathbb{N}$, consider:

$\dfrac{n!}{2^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{n}{2} \rfloor !}$ and $\dfrac{(n+1)!}{2^{\lfloor \frac{n+1}{2} \rfloor} \lfloor \frac{n+1}{2} \rfloor !}$ for $n \in \mathbb{N}$, $n \equiv 1 \pmod 2$.

We must show that these two expressions are equal:

$$\frac{n!}{2^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{n}{2} \rfloor !} = \frac{(n+1)!}{2^{\lfloor \frac{n+1}{2} \rfloor} \lfloor \frac{n+1}{2} \rfloor !}$$

First, multiply both sides by both denominators to give:

$$n!\ 2^{\lfloor \frac{n+1}{2} \rfloor} \lfloor \tfrac{n+1}{2} \rfloor ! = (n+1)!\ 2^{\lfloor \frac{n}{2} \rfloor} \lfloor \tfrac{n}{2} \rfloor !$$

We can then divide both sides by $n!$:

$$2^{\lfloor \frac{n+1}{2} \rfloor} \lfloor \frac{n+1}{2} \rfloor! = (n+1)\ 2^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{n}{2} \rfloor!$$

Since $n \equiv 1 \pmod 2$: $\lfloor \dfrac{n+1}{2} \rfloor = \dfrac{n+1}{2}$ and $\lfloor \dfrac{n}{2} \rfloor = \dfrac{n-1}{2}$.

Therefore, we can re-write the equation as:

$$2^{\frac{n+1}{2}} \frac{n+1}{2}! = (n+1)\ 2^{\frac{n-1}{2}} \frac{n-1}{2}!$$

Since the difference of $\dfrac{n+1}{2}$ and $\dfrac{n-1}{2}$ is 1, we can divide both side of the equation by $\dfrac{n-1}{2}!$ to give:

$$2^{\frac{n+1}{2}} \frac{n+1}{2} = (n+1)\ 2^{\frac{n-1}{2}}$$

$$(\sqrt{2})^{n+1} \frac{n+1}{2} = (n+1)\ (\sqrt{2})^{n-1}$$

Multiplying both sides by $\sqrt{2}$ gives:

$$(\sqrt{2})^{n+2} \frac{n+1}{2} = (n+1)\ (\sqrt{2})^{n}$$

Then dividing both sides by $(\sqrt{2})^n$ gives:

$$2 \frac{n+1}{2} = n+1$$

$$n+1 = n+1$$

Both sides are shown to be identical. Hence, the equation holds true for all odd $n \in \mathbb{N}$.

The graphs below show how the telephone combinations relationship from earlier compares with the maximum connection combinations function (note that the function, $\dfrac{n!}{2^{\frac{n}{2}} \frac{n}{2}!}$, is used so there is no rounding done only so that the relationship can be illustrated).

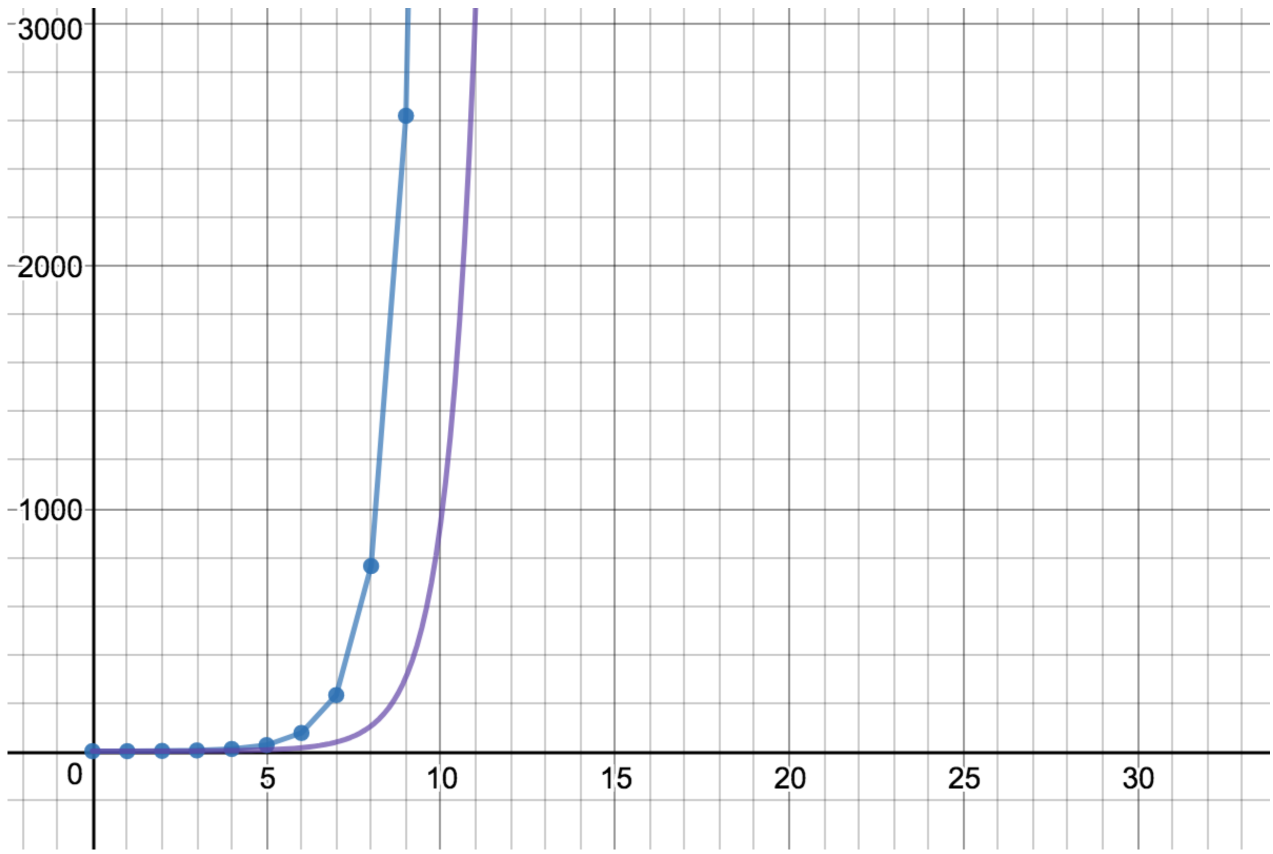


Figure 7: Graphs of telephone connection combinations relation (blue) and factorial maximum connection combinations function (purple)

It can be seen that the slope of the blue graph is steeper than that of the purple graph. This comes as no surprise, since the maximum connection combinations is a subset of the telephone connection combinations. The maximum connection combinations relationship can be used to determine the maximum number of calls which can occur on the phones at once (out of n phones) and to determine how many ways this number of calls can occur between the n people.

6 Conclusion

Finding combinations in different scenarios can be quite a fascinating task and there may be a number solutions for certain situations. The interesting thing about different scenarios is there will always be some method of determining some sort of solution. Different solutions will also use different areas of mathematics, as we have seen in this situation where we used recursion (and recurrence relations) and calculus; both of which are completely different areas of mathematics. Additionally, out of the number of telephone connection combinations of n telephones, it is possible to obtain the maximum number of connections (edges) which can be made and also the number of ways in which this can occur.

As stated, landline telephones have become obsolete in the present day, so there may not be any real purpose of applying this solution now. However, it is still an interesting solution nonetheless and could very well be applied to situations other than connecting telephones.

7 Bibliography

[1] Fannon, Paul *Mathematics Higher Level Topic 10 - Option: Discrete Mathematics for the IB Diploma.*, Cambridge University Press, 2013

[2] Neill, Hugh, and D. A. Quadling. *Mathematics for the IB Diploma Higher Level: Discrete Mathematics.* Cambridge University Press, 2007.

[3] *The Telephone Numbers – Graph Theory.* IB Maths Resources from British International School Phuket, IB Maths Resources, 10 June 2014, `ibmathsresources.com/2014/06/25/the-telephone-numbers-graph-theory/`.